Note: ***For the benefit of the students, specially the aspiring ones, the question of JEE(advanced), 2020 are also given in this booklet. Keeping the interest of students studying in class XI, the questions based on topics from class XI have been marked with '*', which can be attempted as a test. For this test the time allocated in Physics, Chemistry & Mathematics are 30 minutes, 25 minutes and 25 minutes respectively.***

MIITYedu

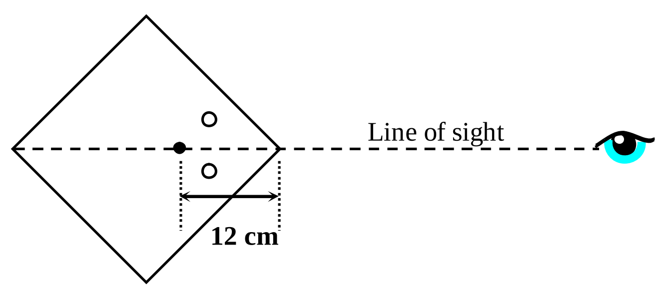
SOLUTIONS TO JEE (ADVANCED) – 2020

PART I: PHYSICS

SECTION 1 (Maximum Marks: 18)

- This section contains **SIX (06)** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER ranging from 0 TO 9, BOTH INCLUSIVE**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : **+3** If ONLY the correct integer is entered;
  Zero Marks : **0** If the question is unanswered;
  Negative Marks : **–1** In all other cases.

Q.1 A large square container with thin transparent vertical walls and filled with water $\left(\text{refractive index} \frac{4}{3}\right)$ is kept on a horizontal table. A student holds a thin straight wire vertically inside the water 12 cm from one of its corners, as shown schematically in the figure. Looking at the wire from this corner, another student sees two images of the wire, located symmetrically on each side of the line of sight as shown. The separation (in cm) between these images is ________.



Sol. **Bonus marks to all students**

*Q.2 A train with cross-sectional area S_t is moving with speed v_t inside a long tunnel of cross-sectional area $S_0(S_0 = 4S_t)$. Assume that almost all the air (density ρ) in front of the train flows back between its sides and the walls of the tunnel. Also, the air flow with respect to the train is steady and laminar. Take the ambient pressure and that inside the train to be p_0. If the pressure in the region between the sides of the train and the tunnel walls is p, then $p_0 - p = \frac{7}{2N}\rho v_t^2$. The value of N is __________.

Sol. **9**

The velocity of air flow relative to the train between its sides and the walls of the tunnel is v

$v \times 3S_t = v_t 4S_t$

$v = \frac{4}{3}v_t$ …(i)

Now,

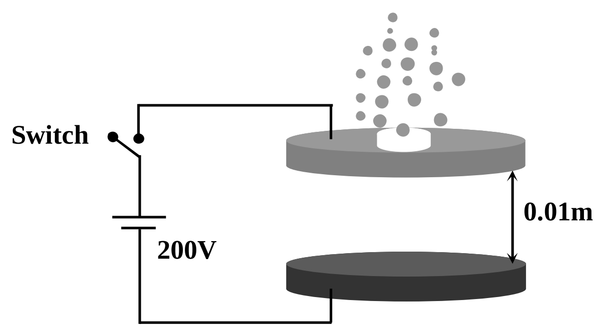
$P + \frac{1}{2}\rho v^2 = P_0 + \frac{1}{2}\rho v_t^2$

$P_0 - P = \frac{1}{2}\rho(v^2 - v_t^2) = \frac{1}{2}\rho v_t^2\left(\frac{16}{9} - 1\right)$

$P_0 - P = \frac{7}{18}\rho v_t^2$

Hence N = 9

Q.3 Two large circular discs separated by a distance of 0.01 m are connected to a battery via a switch as shown in the figure. Charged oil drops of density 900 kg m^{-3} are released through a tiny hole at the center of the top disc. Once some oil drops achieve terminal velocity, the switch is closed to apply a voltage of 200 V across the discs. As a result, an oil drop of radius 8×10^{-7} m stops moving vertically and floats between the discs. The number of electrons present in this oil drop is ________. (neglect the buoyancy force, take acceleration due to gravity = 10 ms^{-2} and charge on an electron (e) = 1.6×10^{-19} C)



Sol. **6**

Electric field between the discs after closing the switch

$E = \frac{V}{d} = \frac{200}{0.01} = 2 \times 10^4$ N/C

When an oil drop stops and floats between the discs

qE = mg

neE = mg

$n = \frac{mg}{eE} = \frac{\rho \frac{4}{3}\pi r^3 g}{eE}$

$n = 900 \times \frac{4 \times 3.14 \times (8 \times 10^{-7})^3 \times 10}{3 \times 1.6 \times 10^{-19} \times 2 \times 10^4}$

$n = \frac{3 \times 4 \times 3.14 \times 2 \times 8}{100} = 6$

The number of electrons present in this oil drop is

n = 6

*Q.4 A hot air balloon is carrying some passengers, and a few sandbags of mass 1 kg each so that its total mass is 480 kg. Its effective volume giving the balloon its buoyancy is V. The balloon is floating at an equilibrium height of 100 m. When N number of sandbags are thrown out, the balloon rises to a new equilibrium height close to 150 m with its volume V remaining unchanged. If the variation of the density of air with height h from the ground is $\rho(h) = \rho_0 e^{-\frac{h}{h_0}}$, where ρ_0 = 1.25 kg m^{-3} and h_0 = 6000 m, the value of N is _________.

Sol. **4**

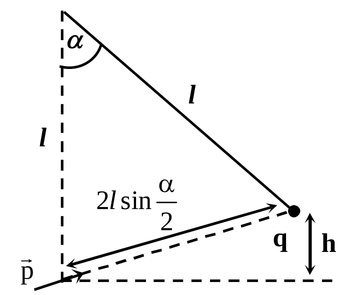
$$\frac{480 - N}{480} = \frac{\rho_2}{\rho_1}$$

$$1 - \frac{N}{480} = e^{-\left(\frac{150-100}{6000}\right)}$$

$$\Rightarrow 1 - \frac{N}{480} = 1 - \frac{5}{600}$$

N = 4

Q.5 A point charge q of mass m is suspended vertically by a string of length l. A point dipole of dipole moment $\vec{p}$ is now brought towards q from infinity so that the charge moves away. The final equilibrium position of the system including the direction of the dipole, the angles and distances is shown in the figure below. If the work done in bringing the dipole to this position is N × (mgh), where g is the acceleration due to gravity, then the value of N is _________ . (Note that for three coplanar forces keeping a point mass in equilibrium, $\frac{F}{\sin\theta}$ is the same for all forces, where F is any one of the forces and θ is the angle between the other two forces)



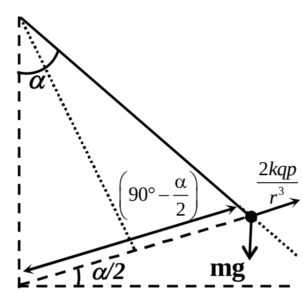
Sol. **2**

$$\frac{mg}{\sin\left(90 + \frac{\alpha}{2}\right)} = \frac{2kqp}{r^3 \sin(180° - \alpha)}$$

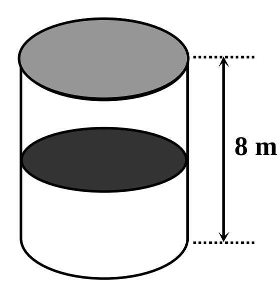
$$\frac{mg\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)} = \frac{2kqp}{r^3}$$

U_{elec}=PE=$\frac{kqp}{r^2} = mgr\sin\left(\frac{\alpha}{2}\right) = mgh$

$$U = U_{grav} + U_{elec} = 2mgh$$



*Q.6 A thermally isolated cylindrical closed vessel of height 8 m is kept vertically. It is divided into two equal parts by a diathermic (perfect thermal conductor) frictionless partition of mass 8.3 kg. Thus the partition is held initially at a distance of 4 m from the top, as shown in the schematic figure below. Each of the two parts of the vessel contains 0.1 mole of an ideal gas at temperature 300 K. The partition is now released and moves without any gas leaking from one part of the vessel to the other. When equilibrium is reached, the distance of the partition from the top (in m) will be _______ (take the acceleration due to gravity = 10 ms^{-2} and the universal gas constant = 8.3 J mol−1K^{-1}).



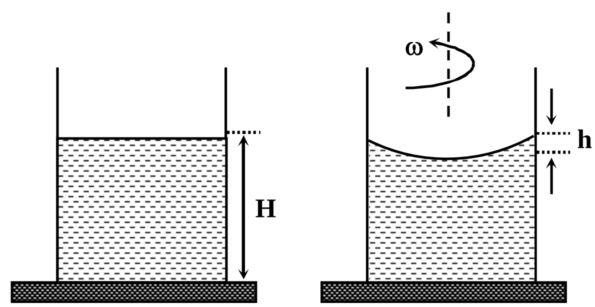
Sol. **Bonus marks to all students**

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If only (all) the correct option(s) is(are) chosen;
Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;
Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : –2 In all other cases.

Q.7 A beaker of radius r is filled with water $\left(\text{refractive index}\frac{4}{3}\right)$ up to a height H as shown in the figure on the left. The beaker is kept on a horizontal table rotating with angular speed ω. This makes the water surface curved so that the difference in the height of water level at the center and at the circumference of the beaker is h(h << H, h << r), as shown in the figure on the right. Take this surface to be approximately spherical with a radius of curvature R. Which of the following is/are correct? (g is the acceleration due to gravity)



(A) $R = \dfrac{h^2 + r^2}{2h}$

(B) $R = \dfrac{3r^2}{2h}$

(C) Apparent depth of the bottom of the beaker is close to $\dfrac{3H}{2}\left(1+\dfrac{\omega^2 H}{2g}\right)^{-1}$

(D) Apparent depth of the bottom of the beaker is close to $\dfrac{3H}{4}\left(1+\dfrac{\omega^2 H}{4g}\right)^{-1}$

Sol. **A, D**

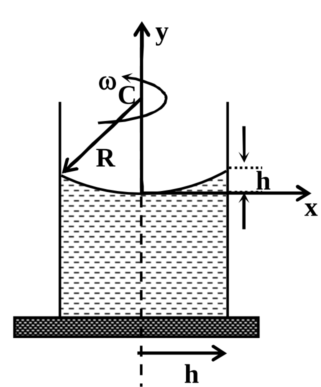
$\tan\theta = \dfrac{dy}{dx} = \dfrac{\omega^2 x}{g}$

$\Rightarrow y = \dfrac{\omega^2 x^2}{2g} \Rightarrow h = \dfrac{\omega^2 r^2}{2g}$

Pythagoras $\Rightarrow r^2 + (R-h)^2 = R^2$

$\Rightarrow R = \dfrac{r^2 + h^2}{2h}$ ⇒ A option is correct

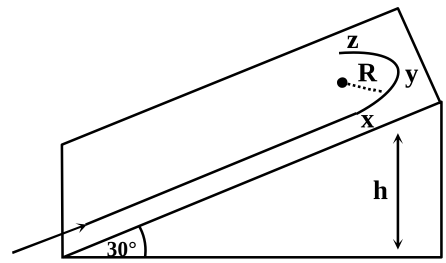
For apparent depth applying refraction formula



$$\frac{1}{v}-\frac{4}{3(H-h)}=\frac{1-\frac{4}{3}}{R}\Rightarrow\left|\frac{1}{v}\right|=\frac{1}{3R}+\frac{4}{3H}\text{ (as H >>h)}$$

$$\Rightarrow |v|=\frac{3H}{4}\left[1+\frac{\omega^2 H}{4g}\right]\Rightarrow \text{D option is correct.}$$

*Q.8 A student skates up a ramp that makes an angle 30° with the horizontal. He/she starts (as shown in the figure) at the bottom of the ramp with speed v_0 and wants to turn around over a semicircular path xyz of radius R during which he/she reaches a maximum height h (at point y) from the ground as shown in the figure. Assume that the energy loss is negligible and the force required for this turn at the highest point is provided by his/her weight only. Then (g is the acceleration due to gravity)



(A) $v_0^2-2gh=\frac{1}{2}gR$

(B) $v_0^2-2gh=\frac{\sqrt{3}}{2}gR$

(C) the centripetal force required at points x and z is zero

(D) the centripetal force required is maximum at points x and z

Sol. **A, D**

Let v be the speed at y

Energy conservation $\Rightarrow \frac{1}{2}mv_0^2=mgh+\frac{1}{2}mv^2$ …(1)

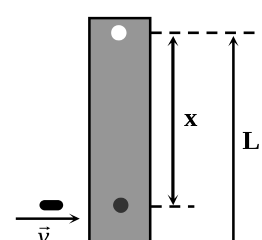
FBD at y $\Rightarrow$ mg sin 30° = $\frac{mv^2}{R}$ …(2)

From (1) and (2)

$\Rightarrow v_0^2-2gh=\frac{gR}{2}$, (A) option is correct.

(D) is true as on the circular path he/she will have maximum speed at x and z.

*Q.9 A rod of mass m and length L, pivoted at one of its ends, is hanging vertically. A bullet of the same mass moving at speed v strikes the rod horizontally at a distance x from its pivoted end and gets embedded in it. The combined system now rotates with angular speed ω about the pivot. The maximum angular speed ω_M is achieved for x = x_M. Then



(A) $\omega=\frac{3vx}{L^2+3x^2}$

(B) $\omega=\frac{12vx}{L^2+12x^2}$

(C) $x_M=\frac{L}{\sqrt{3}}$

(D) $\omega_M=\frac{v}{2L}\sqrt{3}$

Sol. **A, C, D**

Conserving angular momentum about pivot.

$$\Rightarrow mvx = \left(\frac{mL^2}{3} + mx^2\right)\omega$$

$$\Rightarrow \omega = \frac{3vx}{L^2 + 3x^2}$$

For ω to be maximum $\Rightarrow \frac{d\omega}{dx} = 0$,this gives $x = \frac{L}{\sqrt{3}}$ and $\omega_{max} = \frac{\sqrt{3}v}{2L}$

Q.10 In an X-ray tube, electrons emitted from a filament (cathode) carrying current I hit a target (anode) at a distance d from the cathode. The target is kept at a potential V higher than the cathode resulting in emission of continuous and characteristic X-rays. If the filament current I is decreased to $\frac{I}{2}$, the potential difference V is increased to 2V, and the separation distance d is reduced to $\frac{d}{2}$, then

(A) the cut-off wavelength will reduce to half, and the wavelengths of the characteristic X-rays will remain the same
(B) the cut-off wavelength as well as the wavelengths of the characteristic X-rays will remain the same
(C) the cut-off wavelength will reduce to half, and the intensities of all the X-rays will decrease
(D) the cut-off wavelength will become two times larger, and the intensity of all the X-rays will decrease

Sol. **A, C**

$\lambda_C \propto \frac{1}{V}$, current $\propto$ Intensity

Q.11 Two identical non-conducting solid spheres of same mass and charge are suspended in air from a common point by two non-conducting, massless strings of same length. At equilibrium, the angle between the strings is α. The spheres are now immersed in a dielectric liquid of density 800 kg m^{-3} and dielectric constant 21. If the angle between the strings remains the same after the immersion, then

(A) electric force between the spheres remains unchanged
(B) electric force between the spheres reduces
(C) mass density of the spheres is 840 kg m^{-3}
(D) the tension in the strings holding the spheres remains unchanged

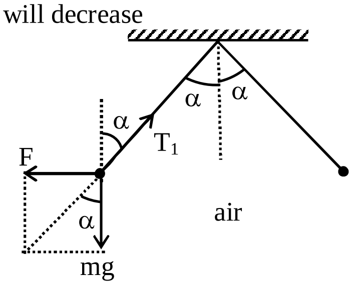
Sol. **A, C**

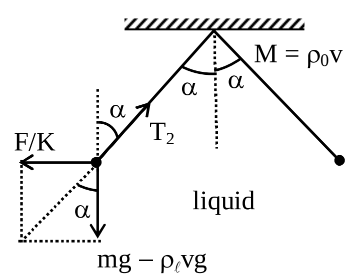
$$\tan\alpha = \frac{F}{mg} = \frac{F/K}{mg - \rho_\ell vg}$$

$\Rightarrow \rho_0 = 840$ kg/m^3

$T_1\cos\alpha = mg,\ T_2\cos\alpha = mg - \rho_\ell vg$

So, T_2 will decrease





*Q.12 Starting at time t = 0 from the origin with speed 1 ms^{-1}, a particle follows a two-dimensional trajectory in the x-y plane so that its coordinates are related by the equation $y = \frac{x^2}{2}$. The x and y components of its acceleration are denoted by a_x and a_y, respectively. Then

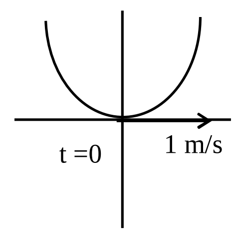
(A) $a_x = 1\ ms^{-2}$ implies that when the particle is at the origin, $a_y = 1\ ms^{-2}$

(B) $a_x = 0$ implies $a_y = 1\ ms^{-2}$ at all times

(C) at t = 0, the particle's velocity points in the x-direction

(D) $a_x = 0$ implies that at t = 1 s, the angle between the particle's velocity and the x axis is 45°

Sol. **A, B, C, D**



$y = \frac{x^2}{2}$

$v_y = xv_x$

$a_y = xa_x + v_x^2$

x = 0 ⇒ $a_y = a_x^2 = 1$ m/s², for any value of a_x

$a_x = 0 \Rightarrow a_y = v_x^2 = 1$

$a_x = 0$

$\tan\theta = \frac{v_x}{v_y} = x = 1$ ($x = v_x t = 1$)

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +4 If ONLY the correct numerical value is entered;
  Zero Marks : 0 In all other cases.

*Q.13 A spherical bubble inside water has radius R. Take the pressure inside the bubble and the water pressure to be p_0. The bubble now gets compressed radially in an adiabatic manner so that its radius becomes (R − a). For a << R the magnitude of the work done in the process is given by $(4\pi p_0 Ra^2)X$, where X is a constant and $\gamma = C_p/C_V = 41/30$. The value of X is________ .

Sol. **2.05 (2.05 to 2.05)**

In adiabatic process

$$dp = -\frac{\gamma p}{V}dV = -\frac{\gamma p_0}{V}(-4\pi R^2 a)$$

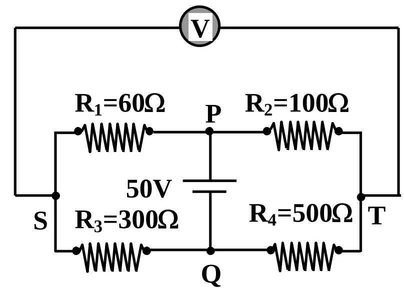
Work done in the process = $-(dp)_{avg}dV = -\frac{dp}{2}dV$

$$= -\frac{\gamma p_0}{2V}(4\pi R^2 a)(-4\pi R^2 a) = (4\pi p_0 Ra^2)\frac{3}{2}\times\frac{41}{30}$$

= (2.05) $(4\pi p_0 Ra^2)$

⇒ x = 2.05

Q.14 In the balanced condition, the values of the resistances of the four arms of a Wheatstone bridge are shown in the figure below. The resistance R_3 has temperature coefficient 0.0004 ℃$^{-1}$. If the temperature of R_3 is increased by 100 ℃, the voltage developed between S and T will be __________ volt.



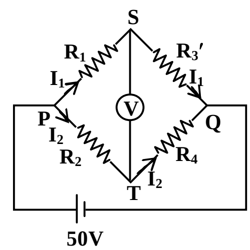
Sol. **0.27 (0.26 to 0.28)**

$R_3' = R_3(1+\alpha\Delta T) = 312\ \Omega$

$V_S - V_T = I_2R_2 - I_1R_1$

$= \dfrac{50}{100+500}\times 100 - \dfrac{50}{60+312}\times 60$

$\approx 0.2688 \approx 0.27$ V



Q.15 Two capacitors with capacitance values C_1 = 2000 ± 10 pF and C_2 = 3000 ± 15 pF are connected in series. The voltage applied across this combination is V = 5.00 ± 0.02 V. The percentage error in the calculation of the energy stored in this combination of capacitors is _______.

Sol. **1.30 (1.30 to 1.30)**

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\Rightarrow \frac{1}{C^2}\Delta C = \frac{1}{C_1^2}\Delta C_1 + \frac{1}{C_2^2}\Delta C_2$$

$$\Rightarrow \frac{\Delta C}{C} = C\left(\frac{\Delta C_1}{C_1^2} + \frac{\Delta C_2}{C_2^2}\right) = = 1200\left(\frac{10}{4\times 10^6} + \frac{15}{9\times 10^6}\right) = 5\times 10^{-3}$$

Energy stored

$$U = \frac{1}{2}CV^2 \Rightarrow \frac{\Delta U}{U} = \frac{\Delta C}{C} + 2\frac{\Delta V}{V}$$

Percentage error in energy stored in the combination of capacitors.

$$\frac{\Delta U}{U}\times 100 = \left(5\times 10^{-3} + 2\frac{0.02}{5}\right)\times 100 = 1.30\%$$

*Q.16 A cubical solid aluminium (bulk modulus = $-V\dfrac{dP}{dV}$ = 70 GPa) block has an edge length of 1 m on the surface of the earth. It is kept on the floor of a 5 km deep ocean. Taking the average density of water and the acceleration due to gravity to be 10^3 kg m^{-3} and 10 ms^{-2}, respectively, the change in the edge length of the block in mm is _____.

Sol. **0.238 (0.23 to 0.25)**

$$B = -V\frac{dP}{dV}$$

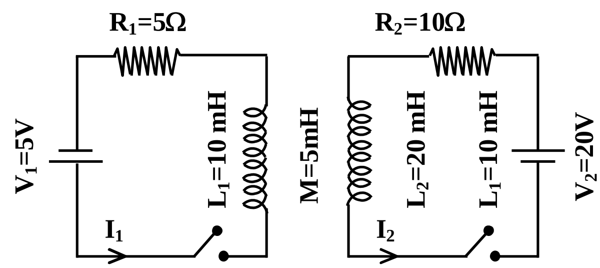
$$70\times 10^9 = \frac{(1)^3\times 5\times 10^3\times 10}{\Delta V}$$

$\Rightarrow \Delta V = \frac{5}{7} \times 10^{-3}$

$\Rightarrow \frac{\Delta V}{V} = \frac{3\Delta \ell}{\ell}$

$\Rightarrow \Delta \ell = \frac{5}{21} = 0.238$ mm

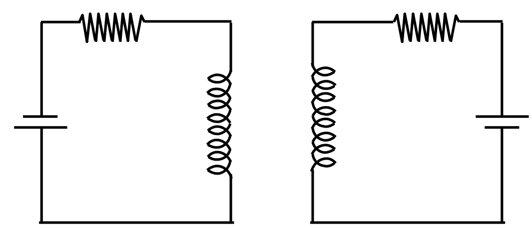
Q.17 The inductors of two LR circuits are placed next to each other, as shown in the figure. The values of the self-inductance of the inductors, resistances, mutual-inductance and applied voltages are specified in the given circuit. After both the switches are closed simultaneously, the total work done by the batteries against the induced EMF in the inductors by the time the currents reach their steady state values is________ mJ.



Sol. **55 (55.00 to 55.00)**

$dU = \phi_1 di_1 + \phi_2 di_2$

$U = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 \pm M I_1 I_2 = 55$ mJ



*Q.18 A container with 1 kg of water in it is kept in sunlight, which causes the water to get warmer than the surroundings. The average energy per unit time per unit area received due to the sunlight is 700 Wm^{-2} and it is absorbed by the water over an effective area of 0.05 m^2. Assuming that the heat loss from the water to the surroundings is governed by Newton's law of cooling, the difference (in °C) in the temperature of water and the surroundings after a long time will be _____________. (Ignore effect of the container, and take constant for Newton's law of cooling = 0.001 s^{-1}, Heat capacity of water = 4200 J kg^{-1} K^{-1})

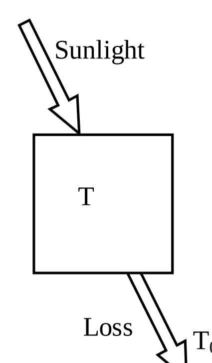
Sol. **8.33 (8.32 to 8.34)**

When steady state is reached energy received per unit time is equal to energy loss per unit time

$700 \times 0.05 = ms\frac{d\Delta T}{dt}$

$\frac{d\Delta T}{dt} = -b\Delta T$

$\Delta T = 8.33$ K



PART II: CHEMISTRY

SECTION 1 (Maximum Marks: 18)

- This section contains SIX (06) questions.
- The answer to each question is a **SINGLE DIGIT INTEGER** ranging from **0** to **9**, **BOTH INCLUSIVE.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer,
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +3 If ONLY the correct integer is entered;
  Zero Marks : 0 If the question is unanswered;
  Negative Marks : -1 In all other cases.

*1. The 1st, 2nd, and the 3rd ionization enthalpies, I_1, I_2, and I_3, of four atoms with atomic numbers n, $n + 1$, $n + 2$, and $n + 3$, where $n < 10$, are tabulated below. What is the value of n?

Atomic Number	Ionization Enthalpy (kJ/mol)		
	I_1	I_2	I_3
n	1681	3374	6050
$n + 1$	2081	3952	6122
$n + 2$	496	4562	6910
$n + 3$	738	1451	7733

Sol. **9**

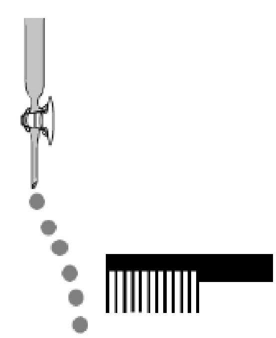
Since I.E. of Na is 495.8 kJ mol^{-1}, so (n + 2) should be 11.

n + 2 = 11, So n = 9

*2. Consider the following compounds in the liquid form:

O_2, HF, H_2O, NH_3, H_2O_2, CCl_4, $CHCl_3$, C_6H_6, C_6H_5Cl.

When a charged comb is brought near their flowing stream, how many of them show deflection as per the following figure?



Sol. **6**

Only polar molecules can be attracted by a charged comb. So, except O_2, CCl_4 and C_6H_6 all can be attracted in electric field.

*3. In the chemical reaction between stoichiometric quantities of $KMnO_4$ and KI in weakly basic solution, what is the number of moles of I_2 released for 4 moles of $KMnO_4$ consumed?

Sol. **6**

In slightly alkaline medium, MnO_4^- reduces to MnO_2 while I^- is oxidized to I_2.

$$MnO_4^- + 2H_2O + 3e^- \longrightarrow MnO_2 + 4OH^- \big) \times 2$$

$$2I^- \longrightarrow I_2 + 2e^-) \times 3$$

$$2MnO_4^- + 6I^- + 4H_2O \longrightarrow 2MnO_2 + 3I_2 + 8OH^-$$

So, 2 mol of $KMnO_4$ can produce 3 mole of I_2

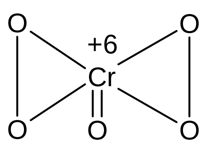
$\therefore$ 4 mol of $KMnO_4$ can produce 6 mole of I_2

*4. An acidified solution of potassium chromate was layered with an equal volume of amyl alcohol. When it was shaken after the addition of 1 mL of 3% H_2O_2, a blue alcohol layer was obtained. The blue color is due to the formation of a chromium (VI) compound '**X**'. What is the number of oxygen atoms bonded to chromium through only single bonds in a molecule of **X**?

Sol. **4**

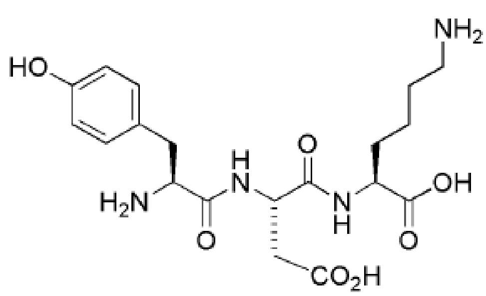
$$CrO_4^{2-} \underset{OH^-}{\overset{H^+}{\rightleftharpoons}} Cr_2O_7^{2-}$$

$$Cr_2O_7^{2-} + H_2O + H^+ \longrightarrow \underset{\substack{\text{Chromium peroxide} \\ (\text{blue})}}{CrO_5}$$



So, number of oxygen atoms attached to Cr through single bond = 4

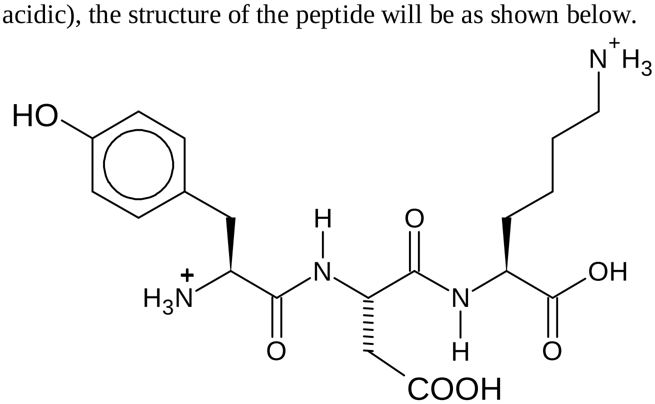
5. The structure of a peptide is given below:



If the absolute values of the net charge of the peptide at pH = 2, pH =6, and pH = 11 are $|z_1|$, $|z_2|$, and $|z_3|$, respectively, then what is $|z_1| + |z_2| + |z_3|$?

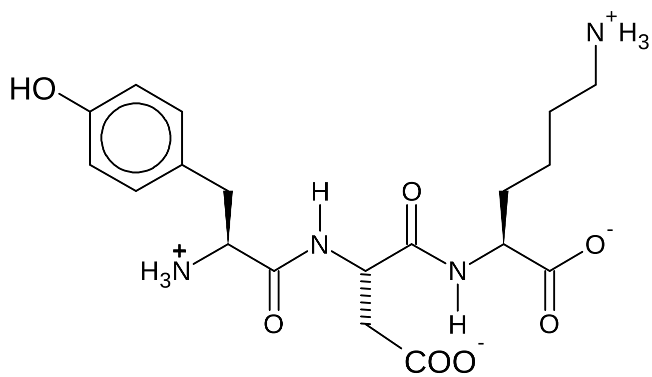
Sol. **5**

At pH = 2 (highly acidic), the structure of the peptide will be as shown below.



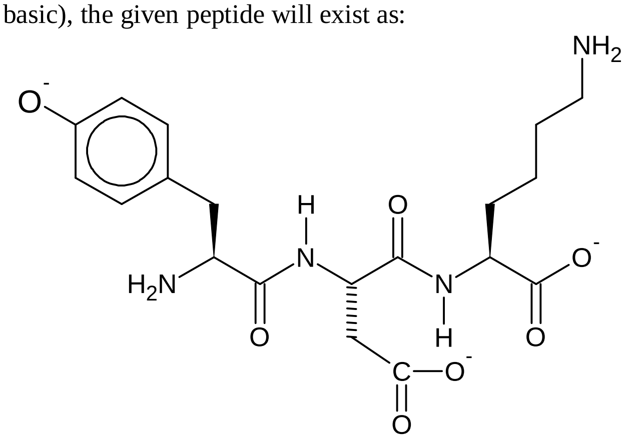
So, $|Z_1| = 2$

At pH = 6 it will exist as neutral or zwitter-ion structure as shown below.



So, according to this condition, $|Z_2| = 0$

At pH = 11 (highly basic), the given peptide will exist as:



So, $|Z_3| = |-3| = 3$

$$|Z_1| + |Z_2| + |Z_3| = 2 + 0 + 3 = 5$$

*6. An organic compound ($C_8H_{10}O_2$) rotates plane-polarized light. It produces pink color with neutral $FeCl_3$ solution. What is the total number of all the possible isomers for this compound?

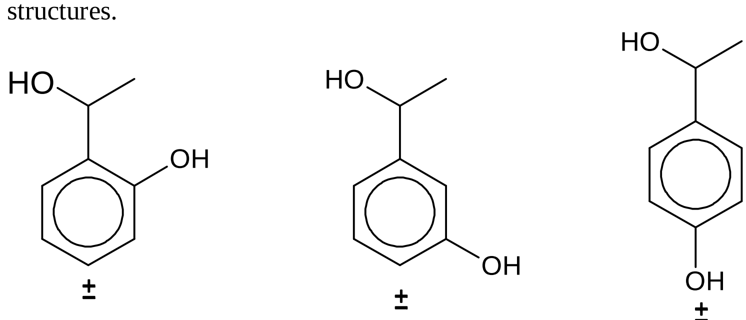
Sol. **6** (Considering that all isomers satisfy the given conditions)

Since given compound produces pink colour with neutral $FeCl_3$.

So, it must have at least one –OH group attached to benzene ring.

Also, given compound is optically active, it means it should have a chiral centre, which is only possible when it has $–CH(OH)CH_3$

So, based on the above observations, compound ($C_8H_{10}O_2$) is expected to possess the following isomeric structures.



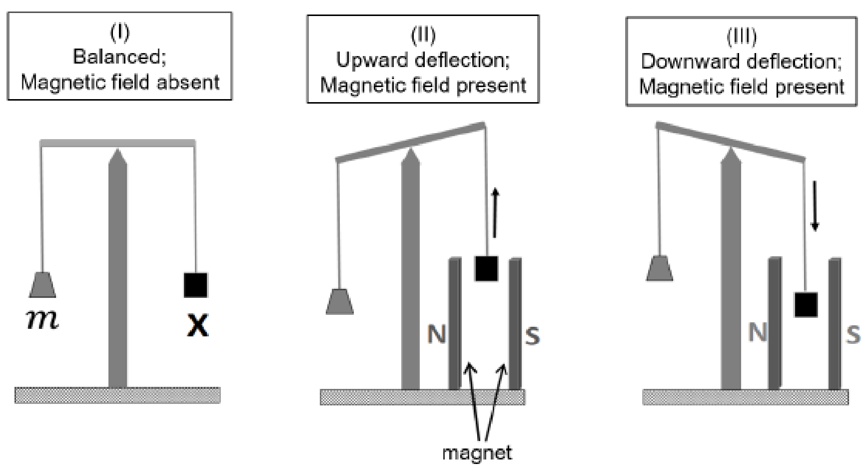
So, total isomeric structures = 6

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks	:	+4 If only (all) the correct option(s) is(are) chosen.
Partial Marks	:	+3 If all the four options are correct but ONLY two options are chosen,
Partial Marks	:	+2 If three or more options are correct but ONLY one options are chosen, both of which are correct.
Partial Marks	:	+1 If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	:	0 If none of the options is chosen (i.e., the question is unanswered);
Negative Marks	:	-2 In all other cases.

*7. In an experiment, *m* grams of a compound **X** (gas/liquid/solid) taken in a container is loaded in a balance as shown in figure **I** below. In the presence of a magnetic filed, the pan with **X** is either deflected upwards (figure **II**), or deflected downwards (figure **III**), depending on the compound **X**. Identify the correct statements(s).



(A) If **X** is $H_2O(l)$, deflection of the pan is upwards.
(B) If **X** is $K_4[Fe(CN)_6]$ (s), deflection of the pan is upwards.
(C) If **X** is $O_2(g)$, deflection of the pan is downwards.
(D) If **X** is $C_6H_6(l)$, deflection of the pan is downwards.

Sol. **A,B,C**

Diamagnetic (all electrons paired) are repelled i.e. magnetized in opposite direction in magnetic field, so such substance are deflected upwards in magnetic field. Diamagnetic substances are

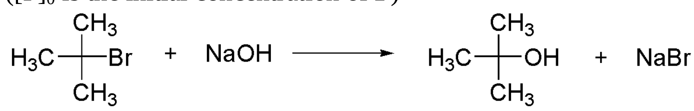
$$H_2O(\ell),\ K_4\left[Fe(CN)_6\right],\ C_6H_6(\ell)$$

Paramagnetic substances (at least one unpaired electron) are attracted in magnetic field, so such substances are deflected downwards.

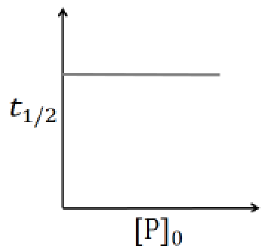
$$O_2(g):\sigma_{1s^2}\ \sigma^*_{1s^2}\ \sigma_{2s^2}\ \sigma_{2p_z^2}\ \pi_{2p_x^2}=\pi_{2p_y^2}\ \pi^*_{2p_x^1}=\pi^*_{2p_y^1}$$

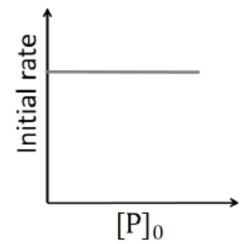
Since O_2 molecules has two unpaired electrons in antibonding π. MO.
So, O_2 is paramagnetic.

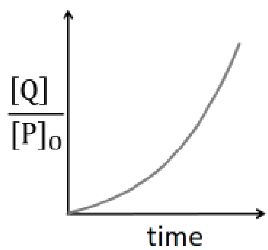
8. Which of the following plots is (are) correct for the given reaction?
($[P]_0$ is the initial concentration of **P**)

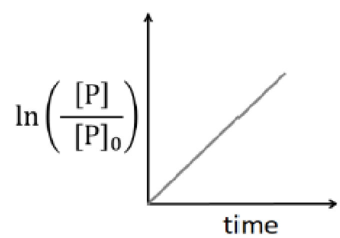


P **Q**

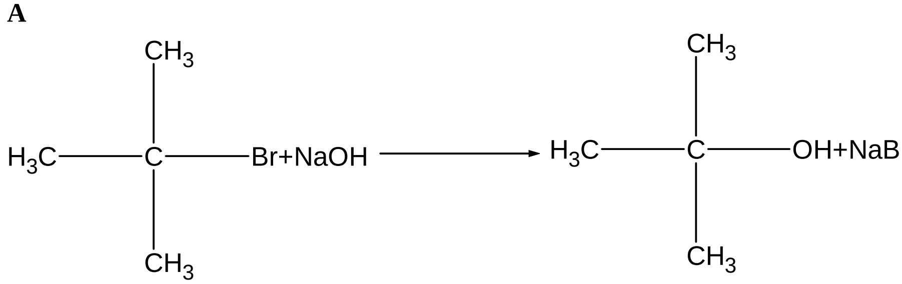
(A) 

(B) 

(C) 

(D) 

Sol. **A**



(P) (Q)

The order of the above reaction is one (S_N1) because substrate is a 3^0-halide

So, for a first order reaction.

$$P = P_0 \times e^{-kt} \text{ or } \frac{P}{P_0} = e^{-kt}$$

$$\ln\left(\frac{P}{P_0}\right) = -kt$$

y = mx

So, the slope of the graph $\ln\left(\frac{P}{P_0}\right)$ versus time (t) should be negative.

Hence option D is incorrect.

Also $\frac{P}{P_0} = e^{-kt}$

$$\left(1-\frac{P}{P_0}\right) = \left(1-e^{-kt}\right)$$

$$\frac{P_0 - P}{P_0} = \left(1-e^{-kt}\right)$$

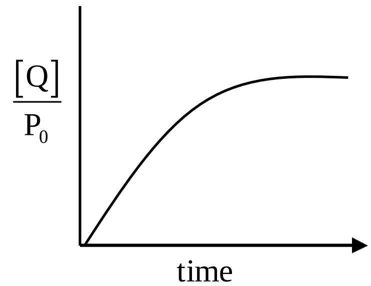


Fig:1

Or $\frac{[Q]}{P_0} = \left(1-e^{-kt}\right)$

So, graph between $\frac{[Q]}{P_0}$ Vs. t, should be as shown in figure (1), so, option (C)is incorrect

For a first order reaction

$r = k[P]^1$

At t = 0, $r_i = k[P_0]$

So, initial rate of reaction is directly proportional to initial conc. of P.

Hence option B is also incorrect.

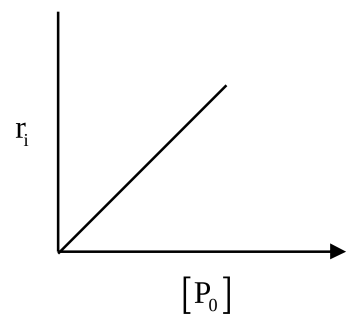


Fig:2

For a first order reaction

$$t_{1/2} = \frac{\ln 2}{k}[P_0]^0$$

So, $t_{1/2}$ is independent of initial conc. of reactant.

So, option (A) is correct.

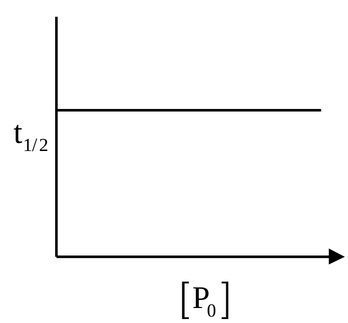


Fig:3

9. Which among the following statement(s) is(are) true for the extraction of aluminium from bauxite?
(A) Hydrated Al_2O_3 precipitates, when CO_2 is bubbled through a solution of sodium aluminate.
(B) Addition of Na_3AlF_6 lowers the melting point of alumina.
(C) CO_2 is evolved at the anode during electrolysis.
(D) The cathode is a steel vessel with a lining of carbon.

Sol. **A,B,C,D**

Sodium meta aluminate is precipitated in the form of $Al(OH)_3$ or $Al_2O_3.xH_2O$. i.e hydrated aluminium hydroxide by passing CO_2 gas.

$$2NaAlO_2 + CO_2(g) + 3H_2O(\ell) \longrightarrow \underset{\text{Or } Al_2O_3.xH_2O}{2Al(OH)_3} \downarrow + Na_2CO_3$$

In electrolytic reduction of Al_2O_3 by Hall-Harcult process, cryolite (Na_3AlF_6) and fluorspar (CaF_2) are added in order to reduce the melting temperature of Al_2O_3.

Anode is made up of graphite,

So, $C(s) + 2O^{2-} \longrightarrow CO_2(g) + 4e^-$

Cathode is a steel vessel lined with graphite.

10. Choose the correct statement(s) among the following

(A) $SnCl_2 . 2H_2O$ is a reducing agent.

(B) SnO_2 reacts with KOH to form $K_2[Sn(OH)_6]$.

(C) A solution of $PbCl_2$ in HCl contains Pb^{2+} and Cl^- ions.

(D) The reaction of Pb_3O_4 with hot dilute nitric acid to give PbO_2 is a redox reaction.

Sol. **A, B**

(A) Sn^{2+} is less stable than Sn^{4+}, therefore $SnCl_2.2H_2O$ is a Lewis acid.

(B) SnO_2 is amphoteric, so it will dissolve in KOH solution forming K_2SnO_3 or $K_2[Sn(OH)_6]$.

$$SnO_2 + 2KOH + 2H_2O \longrightarrow K_2\left[Sn(OH)_6\right]$$

(C) $PbCl_2$ will form a complex with HCl

$$PbCl_2(s) + 2HCl \longrightarrow H_2[PbCl_4]$$

So, we may assume that now this complex will dissociate as:

$$H_2[PbCl_2] \longrightarrow 2H^+ + [PbCl_4]^{2-}$$

So, considering this possibility, option (C) should be incorrect.

(D) Pb_3O_4 reacts with conc.HNO_3 to form PbO_2 but the reaction is not a redox reaction.

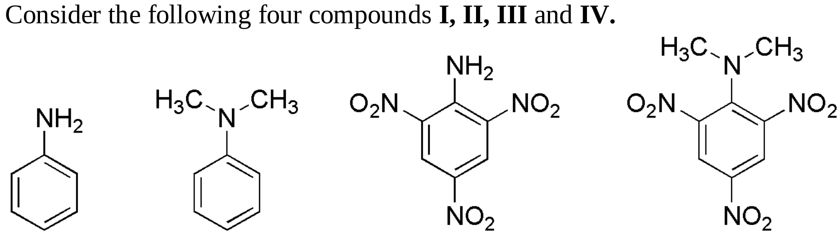
$$\overset{(+4)}{Pb}O_2.2\overset{(+2)}{Pb}O + \underset{(Conc.)}{4HNO_3} \longrightarrow \overset{(+4)}{Pb}O_2 \downarrow + 2\overset{(+2)}{Pb}(NO_3)_2 + 6H_2O$$

or

Pb_3O_4

So, option **(D)** is incorrect.

*11. Consider the following four compounds **I, II, III** and **IV.**



I **II** **III** **IV**

Choose the correct statement(s).

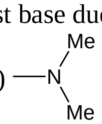
(A) The order of basicity is **II > I > III > IV.**

(B) The magnitude of pK_b difference between **I** and **II** is more than that between **III** and **IV.**

(C) Resonance effect is more in **III** than in **IV.**

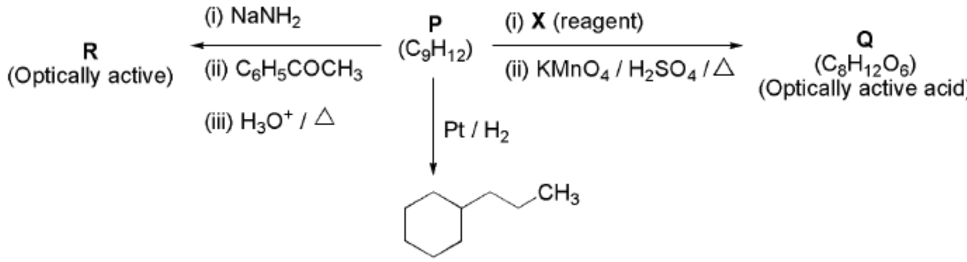
(D) Steric effect makes compound **IV** more basic than **III.**

Sol. **C, D**

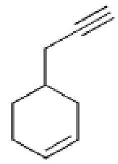
- ❖ (III) is the weakest base due to the presence of three $-NO_2$ group, which exert strong –M effect.
- ❖ In compound (IV)  group is thrown out of the plane of aromatic ring due to steric crowding by $-NO_2$ groups at ortho positions as a result of this lone pair of electron on N – atom does not participate in resonance with aromatic ring and hence it acts as an aliphatic amine. So, (IV) is the strongest base.
- ❖ Correct order of basicity is; III < I < II < IV (in basic character).

- ❖ Difference of pK_a values is highest between (III) and (IV).
- ❖ pK_b; (I) $\rightarrow$ 9.42; (II) $\rightarrow$ 8.94, (IV) $\rightarrow$ 4.60

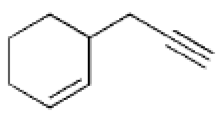
12. Consider the following transformations of a compound **P**.

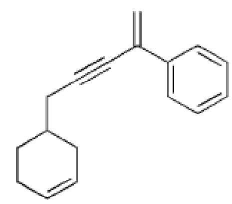


Choose the correct option(s).

(A) **P** is 

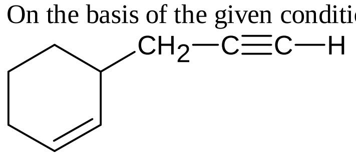
(B) **X** is Pd-C/quinoline/H_2

(C) **P** is 

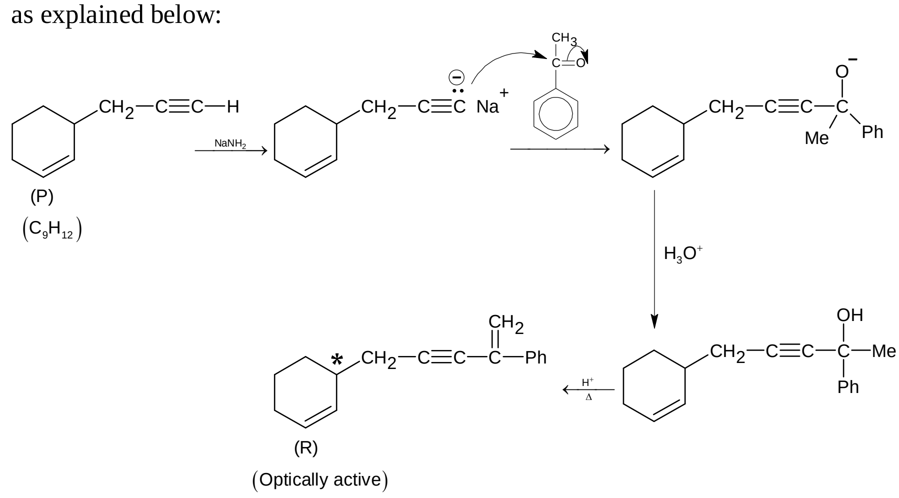
(D) **R** is 

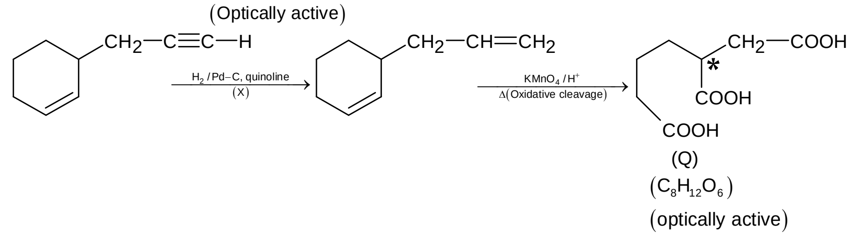
Sol. **B, C**

❖ On the basis of the given conditions, correct structure of (P) should be



as explained below:



❖ 

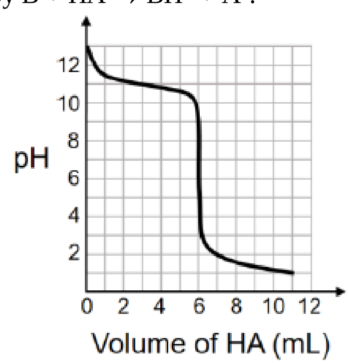
SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer, If the numerical value has more than two decimal places, truncate/round-off the value of **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

Full Marks : +4 If ONLY the correct numerical value is entered;

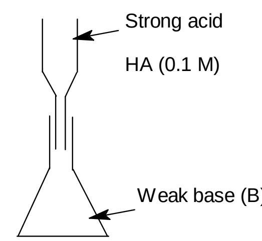
Zero Marks : 0 In all other cases.

*13. A solution of 0.1 M weak base (B) is titrated with 0.1 M of a strong acid (HA). The variation of pH of the solution with the volume of HA added is shown in the figure below. What is the pK_b of the base? The neutralization reaction is given by $B + HA \rightarrow BH^+ + A^-$.



Sol. **03.00**

$$\underset{\text{(weak base)}}{B} + \underset{\text{(strong acid)}}{HA} \longrightarrow BH^+ + A^-$$



❖ From the graph, it is clear, that the equivalence point is reached at a titre value of 6 mL i.e. when 6 mL of HA are added base (B) is completely neutralized. So it will be half neutralized at titre value of 3 mL; when
3 mL of HA is added, 'B' is half – neutralized. So at this stage it will form a best buffer.
i.e. $pOH = pK_b = 3$
So, pK_b of weak base = 3

14. Liquid **A** and **B** form ideal solution for all compositions of **A** and **B** at 25^0C. Two such solutions with 0.25 and 0.50 mole fractions of **A** have the total vapour pressures of 0.3 and 0.4 bar, respectively. What is the vapour pressure of pure liquid **B** in bar?

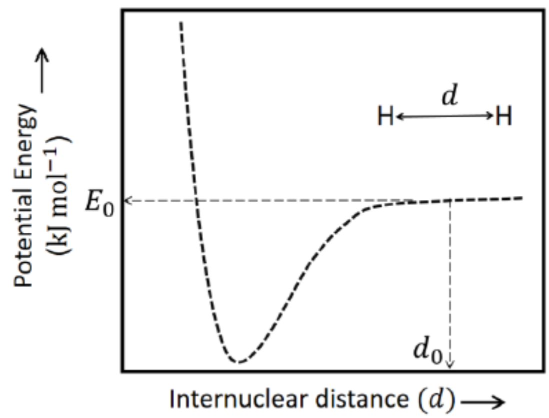
Sol. **0.20**

$$0.3 = p_A^0 \times 0.25 + p_B^0 \times 0.75 \quad ...(1)$$

$$0.4 = p_A^0 \times 0.50 + p_B^0 \times 0.50 \quad ...(2)$$

On solving, p_B^0 = 0.20 bar and p_A^0 = 0.6 bar

*15. The figure below is the plot of potential energy versus internuclear distance (d) of H_2 molecule in the electronic ground state. What is the value of the net potential energy E_0 (as indicated in figure) in kJ mol^{-1}, for $d = d_0$ at which the electron-electron repulsion and the nucleus-nucleus repulsion energies are absent? As reference, the potential energy of H atom is taken as zero when its electron and the nucleus are infinitely far apart.
Use Avogadro constant as 6.023×10^{23} mol^{-1}.



Sol. **–5246.50**

$$P.E. = -\frac{Kq_1q_2}{r}$$

P.E. of one H – atom in ground state = $-\frac{K(e)(e)}{r}$

$$P.E. = \frac{-9\times10^9\times(1.6\times10^{-19})^2}{0.529\times10^{-10}} = -43.5538\times10^{-19}\ J/atom$$

So, P.E. of two H – atoms (or one molecules of H_2) = $-2 \times 43.5538 \times 10^{-19}$ = -87.106×10^{-19} J/molecule

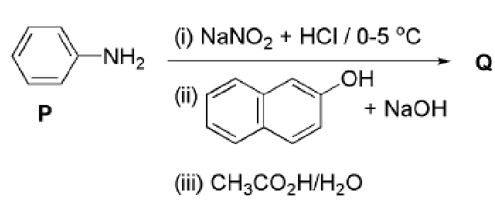
So, P.E. of one mole of H_2 molecules = $-87.106 \times 10^{-19} \times 6.023 \times 10^{23}$ J

$= -524.6490 \times 10^4$

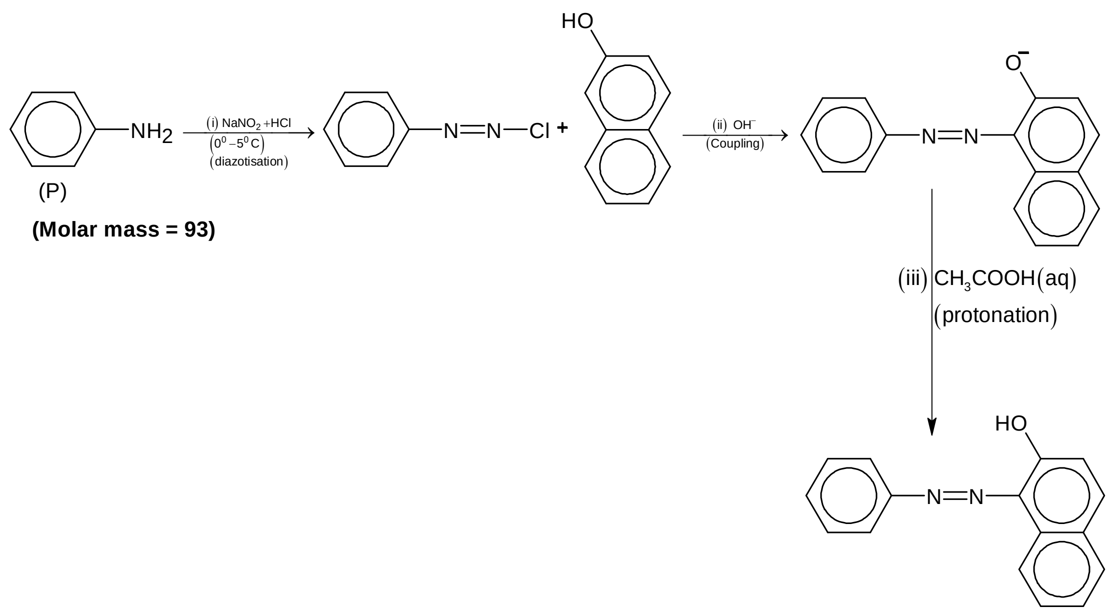
$= -5246.49$ kJ/mol^{-1}

≈ -5246.50 kJmol^{-1}

16. Consider the reaction sequence from **P** to **Q** shown below. The overall yield of the major product **Q** from **P** is 75%. What is the amount in grams of **Q** obtained from 9.3 mL of **P**? (Use density of **P** = 1.00 g mL^{-1}; Molar mass of C = 12.0, H = 1.0, O = 16.0 and N = 14.0 g mol^{-1})



Sol. **18.60**



(P)

(Molar mass = 93)

(Q)

(Molar mass = 248)

Moles of P = $\frac{9.3}{93} = 0.1 = \text{moles of Q.}$

So, mass of (Q) = 0.1 × 248 × 0.75 = 18.60 gram

*17. Tin is obtained from cassiterite by reduction with coke. Use the data given below to determine the minimum temperature (in K) at which the reduction of cassiterite by coke would take place.
At 298 K; $\Delta_f H^0$ (SnO_2 (*s*)) = –581.0 kJ mol^{-1}, $\Delta_f H^0$ ($CO_2(g)$) = – 394.0 kJ mol^{-1},
S^0 (SnO_2 (*s*)) = 56.0 J K^{-1} mol^{-1}, S^0 (Sn(*s*))= 52.0 J K^{-1} mol^{-1},
S^0 (C (*s*)) = 6.0 J K^{-1} mol^{-1}, S^0 ($CO_2(g)$)= 210.0 J K^{-1} mol^{-1},
Assume that the enthalpies and the entropies are temperature independent.

Sol. **935**

$$SnO_2(s) + C(s) \longrightarrow Sn(s) + CO_2(g)$$

$$\Delta H_r^0 = -394.0 + 581 = 187\ kJmol^{-1}$$

$$\Delta S^0 = 210 + 52 - 6 - 56 = 200\ Jmol^{-1}K^{-1}$$

Equilibrium temperature $(T)_{eq} = \frac{187 \times 1000}{200} = 935\ K$

So, T > 935 K (for spontaneity)

*18. An acidified solution of 0.05 M Zn^{2+} is saturated with 0.1 M H_2S. What is the minimum molar concentration (M) of H^+ required to prevent the precipitation of ZnS?
Use K_{sp} (ZnS) = 1.25 × 10^{-22} and overall dissociation constant of H_2S. $K_{NET} = K_1K_2 = 1 \times 10^{-21}$

Sol. **0.20**

$$ZnS(s) \rightleftharpoons Zn^{2+}(aq) + S^{2}(aq)$$

$$\left[S^{2-}\right] = \frac{K_{sp}}{\left[Zn^{2+}\right]} = \frac{1.25\times10^{-22}}{0.05} = 25\times10^{-22}$$

$$H_2S \rightleftharpoons 2H^{+} + S^{2-}$$

$$10^{-21} = \frac{\left[H^{+}\right]^{2}\times25\times10^{-22}}{0.1}$$

$$\left[H^{+}\right]^{2} = \frac{10^{-22}\times10^{22}}{25} = \frac{1}{25}$$

$$\left[H^{+}\right] = \frac{1}{5} = 0.20\text{ M}$$

PART III: MATHEMATICS

SECTION 1 (Maximum Marks: 18)

- This section contains **SIX (06)** questions.
- The answer to each question is a **SINGLE DIGIT INTEGER ranging from 0 TO 9, BOTH INCLUSIVE**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated *according to the following marking scheme:*
  Full marks : +3 If ONLY the correct integer is entered;
  Zero Marks : 0 If the question is unanswered;
  Negative Marks : –1 In all other cases.

*Q.1. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4iz^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $Re(z_1) > 0$ and $Re(z_2) < 0$, is ______

Sol. **8**

$z^4 - |z|^4 = 4i z^2$

$z^4 - z^2 \cdot \bar{z}^2 = 4iz^2$

$\Rightarrow$ $z^2 = 0$ or $z^2 - \bar{z}^2 = 4i$

Let $z = x + iy$

$\Rightarrow$ $(z+\bar{z})(z-\bar{z}) = 4i$

$\Rightarrow$ $2x \cdot 2iy = 4i$

$\Rightarrow$ $xy = 1$

z_1 lies on $xy = 1$ in first quadrant and z_2 lies on $xy = 1$ in third quadrant.

$\Rightarrow$ $|z_1 - z_2|^2$ is minimum when $z_1 \equiv (1, 1)$ and $z_2 = (-1, -1)$

$\Rightarrow$ $|z_1 - z_2|^2 = 8$

Q.2. The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is NOT less than 0.95, is _____

Sol. **6**

Let ‘n’ be the number of missiles fired

$\Rightarrow$ probability of atleast 3 successful hits

$$= 1 - {}^nC_0\left(\frac{1}{4}\right)^n - {}^nC_1\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{n-1} - {}^nC_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{n-2}$$

$$\Rightarrow 1 - {}^nC_0\left(\frac{1}{4}\right)^n - {}^nC_1\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^{n-1} - {}^nC_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{n-2} \ge 0.95$$

$$\Rightarrow 9n^2 - 3n + 2 \le \frac{4^n}{10}$$

$\Rightarrow$ minimum value of ‘n’ is ‘6’ ($n \in N$)

*Q.3. Let O be the centre of the circle $x^2 + y^2 = r^2$, where $r > \frac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x + 4y = 5$. If the centre of the circumcircle of the triangle OPQ lies on the line $x + 2y = 4$, then the value of r is _____

Sol. **2**

Let R(h, k) be point of intersection of tangents at P and Q on $x^2 - y^2 = r^2$

$\Rightarrow$ equation of chord of contact PQ is $xh + yk = r^2$

Which is also $2x + 4y = 5$

$\Rightarrow$ $(h, k) \equiv \left(\frac{2r^2}{5}, \frac{4r^2}{5}\right)$

Mid-point of OR is circum-centre of ΔOPQ

$\Rightarrow$ $\left(\frac{r^2}{5}, \frac{2r^2}{5}\right)$ lies on $x + 2y = 4 \Rightarrow r = 2$

Q.4. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2×2 matrix such that the trace of A is 3 and the trace of A^3 is – 18, then the value of the determinant of A is _____

Sol. **5**

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, a + d = 3$

$\Rightarrow$ $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

$\Rightarrow$ $A^3 = \begin{bmatrix} a^3 + 2abc + bcd & a^2b + abd + b^2c + bd^2 \\ a^2c + acd + bc^2 + cd^2 & abc + 2bcd + d^3 \end{bmatrix}$

$\Rightarrow$ $a^3 + d^3 + 3abc + 3bcd = -18$

$\Rightarrow$ $(a + d)(a^2 + d^2 - ad) + 3bc(a + d) = -18$

$\Rightarrow$ $(a + d)((a + d)^2 - 3ad) + 3bc(a + d) = -18$

$\Rightarrow$ $3(9 - 3ad) + 9bc = -18$

$\Rightarrow$ $ad - bc = 5$

$\Rightarrow$ $|A| = 5$

Q.5. Let the functions $f:(-1, 1) \to \mathbb{R}$ and $g: (-1, 1) \to (-1, 1)$ be defined by

$f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$,

where [x] denotes the greatest integer less than or equal to x. Let $fog :(-1, 1) \to \mathbb{R}$ be the composite function defined by $(fog)(x) = f(g(x))$. Suppose c is the number of points in the interval $(-1,1)$ at which fog is **NOT** continuous, and suppose d is the number of points in the interval $(-1,1)$ at which fog is **NOT** differentiable. Then the value of c + d is _____

Sol. **4**

$f(x) = |2x - 1| + |2x + 1|$

$g(x) = \{x\}$

$$f(g(x)) = \begin{cases} 2 & -1 < x < -\frac{1}{2} \\ 4x + 4 & -\frac{1}{2} \le x < 0 \\ 2 & 0 \le x < \frac{1}{2} \\ 4x & \frac{1}{2} \le x < 1 \end{cases}$$

$\Rightarrow$ Dis-continuous at $x = 0 \Rightarrow c = 1$

$\Rightarrow$ non-differentiable at $x = -\frac{1}{2}, 0, \frac{1}{2} \Rightarrow d = 3$

$\Rightarrow$ $c + d = 4$

Q.6. The value of the limit

$$\lim_{x\to\frac{\pi}{2}} \frac{4\sqrt{2}\left(\sin 3x+\sin x\right)}{\left(2\sin 2x\sin\frac{3x}{2}+\cos\frac{5x}{2}\right)-\left(\sqrt{2}+\sqrt{2}\cos 2x+\cos\frac{3x}{2}\right)}$$

is _____

Sol. **8**

$$\lim_{x\to\frac{\pi}{2}}\left(\frac{4\sqrt{2}\cdot 2\sin 2x\cos x}{2\sin 2x\sin\frac{3x}{2}+\left(\cos\frac{5x}{2}-\cos\frac{3x}{2}\right)-\sqrt{2}\left(1+\cos 2x\right)}\right)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\left(\frac{16\sqrt{2}\sin x\cos^2 x}{8\sin x\sin\frac{x}{2}\cdot\cos^2 x-2\sqrt{2}\cos^2 x}\right)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x}{8\sin x\sin\frac{x}{2}-2\sqrt{2}}=8$$

SECTION 2 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- The answer has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all)the answer(s).
- Answer to each question will be evaluated *according to the following marking scheme:*

Full marks	:	*+4*	*If only (all) the correct option(s) is(are) chosen;*
Partial Marks	:	*+3*	*If all the four options are correct but ONLY three options are chosen;*
Partial Marks	:	*+2*	*If three or more options are correct but ONLY two options are chosen, both of which are correct;*
Partial Marks	:	*+1*	*If two or more options are correct but ONLY one option is chosen and it is a correct option;*
Zero Marks	:	*0*	*If none of the options is chosen (i.e. the question is unanswered);*
Negative Marks	:	*–2*	*In all other cases.*

Q.7. Let b be a nonzero real number. Suppose f L $\mathbb{R}\rightarrow\mathbb{R}$ is a differentiable function such that f(0) = 1. If the derivative f′ of f satisfies the equation

$$f'(x)=\frac{f(x)}{b^2+x^2}$$

for all $x\in\mathbb{R}$, then which of the following statements is/are TRUE?

(A) If b > 0, then f is an increasing function

(B) If b < 0, then f is a decreasing function

(C) f(x) f(–x) = 1 for all $x\in\mathbb{R}$

(D) f(x) – f(–x) = 0 for all $x\in\mathbb{R}$

Sol. **A, C**

$$f'(x)=\frac{f(x)}{b^2+x^2}$$

$$\Rightarrow \int\frac{f'(x)dx}{f(x)}=\int\frac{dx}{b^2+x^2}\Rightarrow \ell n\left|f(x)\right|=\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+c$$

As $f(0) = 1 \Rightarrow \ell n|1| = 0 + c \Rightarrow c = 0$

$\Rightarrow \quad f(x) = e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$\Rightarrow \quad f(x) . f(-x) = 1$, Also f(x) is increasing $\forall$ b $\in$ R.

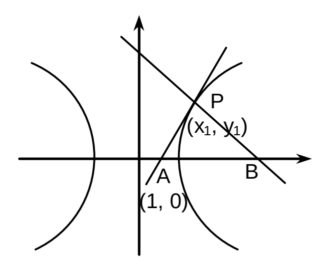
*Q.8. Let a and b be positive real numbers such that a > 1 and b < a. Let P be a point in the first quadrant that lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Suppose the tangent to the hyperbola at P passes through the point (1, 0), and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE?

(A) $1 < e < \sqrt{2}$ (B) $\sqrt{2} < e < 2$

(C) $\Delta = a^4$ (D) $\Delta = b^4$

Sol. **A, D**

As in first quadrant if normal at P is making equal intercepts on axes, then slope of the normal = – 1



$\Rightarrow$ slope of tangent = 1

$\Rightarrow$ equation of tangent at P : $\frac{y-0}{x-1} = 1$ and equation of tangent at (x_1, y_1) : $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

$\Rightarrow x_1 = a^2$ and $y_1 = b^2 \Rightarrow P(x_1, y_1) \equiv P(a^2, b^2)$

$\Rightarrow$ equation of normal at P: $x + y = a^2 + b^2$

$\Rightarrow B \equiv (a^2 + b^2, 0)$

$\Rightarrow \Delta = \frac{1}{2} \times b^2 \times (a^2 + b^2 - 1) = \frac{1}{2} \times b^2 \times 2b^2 = b^4 \Rightarrow \Delta = b^4$

$\left(\because (a^2, b^2) \text{ lies on } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow a^2 - b^2 = 1\right)$

Also $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{a^2 + b^2}{a^2}}$

$\Rightarrow e = \sqrt{\frac{2a^2 - 1}{a^2}} \Rightarrow e = \sqrt{2 - \frac{1}{a^2}} \Rightarrow 1 < e < \sqrt{2}$

$\Rightarrow e \in (1, \sqrt{2})$

Q.9. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions satisfying

$f(x + y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$

for all x, y $\in \mathbb{R}$. If $\lim_{x \to 0} g(x) = 1$, then which of the following statements is/are TRUE?

(A) f is differentiable at every $x \in \mathbb{R}$

(B) If g(0) = 1, then g is differentiable at every $x \in \mathbb{R}$

(C) The derivative f′(1) is equal to 1

(D) The derivative f′(0) is equal to 1

Sol. **A, B, D**

$f(x) = x . g(x) \; \forall \; x \in R$

$\Rightarrow \quad \lim_{x\to 0} f(x) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} g(x)\right)$

$\Rightarrow \quad \lim_{x\to 0} f(x) = 0$

Also $\lim_{x\to 0}\left(f(x+y)\right) = \lim_{x\to 0}\left(f(x)+f(y)+f(x)\cdot f(y)\right)$

$\Rightarrow \quad \lim_{x\to 0} f(x+y) = f(y)$

$\Rightarrow$ f(x) is continuous $\forall\, x \in R$

$\Rightarrow$ f(0) = 0

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{f(h)+f(x)\cdot f(h)}{h}$

$\Rightarrow \quad f'(x) = \lim_{h\to 0}\frac{f(h)}{h}\cdot\left(1+f(x)\right)$

$\Rightarrow \quad \int\frac{f'(x)dx}{1+f(x)} = \int 1\,dx$

$\Rightarrow \quad \ell n\left|1+f(x)\right| = x + c$

f(0) = 0 $\Rightarrow$ c = 0

$\Rightarrow$ $f(x) = e^x - 1$

$\Rightarrow$ $f'(x) = e^x$

$\Rightarrow$ f(x) is differentiable and f′(0) = 1

Also $g(x) = \frac{f(x)}{x}, x \neq 0$

Now if g(0) = 1

$\Rightarrow \quad g(x) = \begin{cases} \frac{e^x-1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ $\Rightarrow$ g(x) is continuous at x = 0

$\Rightarrow \quad g'(0) = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{\frac{e^h-1}{h}-1}{h} = \frac{1}{2}$

$\Rightarrow$ g(x) is diff. $\forall\, x \in R$ if g(0) = 1

Q.10. Let α, β, γ, δ be real numbers such that $\alpha^2 + \beta^2 + \gamma^2 \neq 0$ and α + γ = 1. Suppose the point (3, 2, −1) is the mirror image of the point (1, 0, −1) with respect to the plane αx + βy + γz = δ. Then which of the following statements is/are TRUE?

(A) α + β = 2

(B) δ − γ = 3

(C) δ + β = 4

(D) α + β + γ = δ

Sol. **A, B, C**

Direction ratio of line joining (3, 2, –1) & (1, 0, –1) is (2, 2, 0) proportional to (α, β, γ)

$\Rightarrow$ γ = 0, α + γ = 1, α = 1, β = 1

So plane x + y = δ (2, 1, –1) satisfies if δ = 3

Q.11. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ} = a\hat{i} + b\hat{j}$ and $\overrightarrow{PS} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram PQRS. Let $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w} = \hat{i} + \hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, respectively. If $|\vec{u}| + |\vec{v}| = |\vec{w}|$ and if the area of the parallelogram PQRS is 8, then which of the following statements is/are TRUE?

(A) a + b = 4
(B) a – b = 2
(C) The length of the diagonal PR of the parallelogram PQRS is 4
(D) $\vec{w}$ is an angle bisector of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$

Sol. **A, C**

$$\vec{u} = \frac{\left(a\hat{i} + b\hat{j}\right)\cdot\left(\hat{i} + \hat{j}\right)\widehat{PQ}}{\sqrt{a^2 + b^2}}, \quad \vec{v} = \frac{\left(a\hat{i} - b\hat{j}\right)\cdot\left(\hat{i} + \hat{j}\right)\widehat{PS}}{\sqrt{a^2 + b^2}}$$

Given $|\vec{u}| + |\vec{v}| = |\vec{w}|$

$\Rightarrow \quad |a + b| + |a - b| = \sqrt{2\left(a^2 + b^2\right)}$

$= (2a)^2 = 2(a^2 + b^2)$

$a^2 = b^2$ $\qquad$ $a \geq b$ or $a \leq b$

Area $a^2\left|\left(\hat{i} + \hat{j}\right)\times\left(\hat{i} - \hat{j}\right)\right| = 8$, a = 2

Diagonal 4i or 4j

*Q.12. For nonnegative integers s and r, let

$$\binom{s}{r} = \begin{cases} \dfrac{s!}{r!(s-r)!} & \text{if } r \leq s, \\ 0 & \text{if } r > s. \end{cases}$$

For positive integers m and n, let

$$g(m, n) - \sum_{p=0}^{m+n} \frac{f(m,n,p)}{\binom{n+p}{p}}$$

where for any nonnegative integer p,

$$f(m,n,p) = \sum_{i=0}^{p} \binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}$$

Then which of the following statements is/are TRUE?

(A) g(m, n) = g(n, m) for all positive integers m, n
(B) g(m, n + 1) = g(m + 1, n) for all positive integers m, n
(C) g(2m, 2n) = 2g(m, n) for all positive integers m, n
(D) $g(2m, 2n) = (g(m,n))^2$ for all positive integers m, n

Sol. **A, B, D**

$f(m,n,p) = \sum_{i=0}^{p}\left({}^{m}C_i \; {}^{n+i}C_p \; {}^{p+n}C_{p-i}\right)$

$$= \sum_{i=0}^{p}\left(\frac{m!}{i!(m-1)!}\cdot\frac{(n+i)!}{p!(n+i-p)!}\cdot\frac{(p+n)!}{(p-i)!(n+i)!}\right) = \sum_{i=0}^{p} {}^{m}C_i \cdot {}^{n+p}C_n \cdot {}^{n}C_{p-i}$$

$= {}^{n+p}C_n \cdot {}^{m+n}C_p$

$$g(m_1n) = \sum_{p=0}^{m+n}\frac{{}^{m+n}C_p \cdot {}^{n+p}C_n}{{}^{n+p}C_p} = \sum_{p=0}^{m+n} {}^{m+n}C_p$$

$= 2^{m+n}$.

Section 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal place, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated <u>*according to the following marking scheme:*</u>
  Full marks : +4 If ONLY the correct numerical value is entered;
  Zero Marks : 0 In all other cases.

*Q.13. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that no two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is _____

Sol. **495.00**
We need to put 4 stick (identical) between 11 balls (identical) such that no 2 stick are together
$^{12}C_4 = 495$

*Q.14. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is _____

Sol. **1080.00**
Two rooms accommodate two persons each other two room have one person each.
Total ways $\frac{6!}{(2!)^2(1!)^2\,2!2!} \times 4! = 1080$

Q.15. Two fair dice, each with faces numbered 1, 2, 3, 4, 5 and 6, are rolled together and the sum of the numbers on the faces is observed. This process is repeated till the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If p is the probability that this perfect square is an odd number, then the value of 14p is _____

Sol. **8.00**
Sum is perfect square if die's show {(1, 3)(3, 1)(2, 2)(3, 6)(6, 3)(4, 5)(5, 4)} so probability sum is 9 when it is perfect square p = 4/7
14p = 8

Q.16. Let the function $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \frac{4^x}{4^x + 2}.$$

Then the value of

$$f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \ldots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right)$$

is _____

Sol. **19.00**

$$\sum_{r=1}^{39} f\left(\frac{r}{40}\right) = \frac{1}{2}\sum_{r=1}^{39} f\left(\frac{r}{40}\right) + f\left(\frac{40-r}{40}\right) = \frac{39}{2}$$

As (f(x) + f(1 – x) = 1) & $f\left(\frac{1}{2}\right) = \frac{1}{2}$

So $\sum_{r=1}^{39} f\left(\frac{r}{40}\right) - f\left(\frac{1}{2}\right) = \frac{39}{2} - \frac{1}{2}$ = 19

Q.17. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$. If $F: [0, \pi] \to \mathbb{R}$ is defined by $F(x) = \int_0^x f(t)dt$, and if

$$\int_0^\pi \left(f'(x) + F(x)\right) \cos x \, dx = 2$$

then the value of f(0) is _____

Sol. **4.00**

$$\int_0^\pi \left(f'(x) + F(x)\right) \cos x \, dx$$

$$\int_0^\pi \left[F''(x)\cos x - F'(x)\sin x + F'(x)\sin x + F(x)\cos x\right] dx$$

$$= \int_0^\pi \left(F'(x)\cos x + F(x)\sin x\right)$$

$$= -F'(\pi) - F'(0) = -f'(\pi) - f(0) = 2$$

$f(0) = 4$

Q.18. Let the function $f:(0, \pi) \to \mathbb{R}$ be defined by

$$f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4.$$

Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1\pi, ..., \lambda_r\pi\}$, where $0 < \lambda_1 < \ldots < \lambda_r < 1$. Then the value of $\lambda_1 + \ldots + \lambda_r$ is _____

Sol. **0.5**

$f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4$

$f'(\theta) = 2(\sin\theta + \cos\theta)(\cos\theta - \sin\theta) + 4(\sin\theta - \cos\theta)^3 (\cos\theta + \sin\theta)$

$= 2(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)(1 - 2\sin2\theta)$

$f'(\theta) = 0 \quad \theta = \dfrac{\pi}{12}, \dfrac{\pi}{4}, \dfrac{5\pi}{12}, \dfrac{3\pi}{4}$

– + – + –

0 π/12 π/4 5π/12 3π/4 π

Pts of minima at $x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}$

$$\lambda_1 + \lambda_2 = \frac{1}{12} + \frac{5}{12} = \frac{1}{2} = 0.5$$

*****